# Topical Truthmaker Semantics

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**Abstract:** I present a new kind of truthmaker semantics – topical truthmaker semantics – in its exact and inexact versions with and without restrictions, allowing me to prove soundness and completeness with respect to a number of logics that preserve both truth and aboutness, namely AL, WK3, LP and CL. These logics in turn have applications for the characterization of how agents with different informational states should update their belief systems. Topical truthmaker semantics thus allows for a new conception of subject matters that does not need to equate subject matters with impossible states.
**Keywords:** truthmaker semantics; subject matters; Weak Kleene; questions under discussion.

## 1 Introduction

Truthmaker semantics has been flourishing and there have been a number of fascinating applications of it to various philosophical issues in epistemic logic (Hawke and Özgün, forthcoming), in deontic logic (Fine, 2018; Anglberger, Korbmacher and Faroldi, 2016) as well as in characterizing certain notions such as verisimilitude (Fine, 2019). One prominent application has been to the notion of topic or subject matter (Yablo, 2014; Fine, 2017, 2020), where the subject matter of a sentence has been influentially characterized by Fine (2020) as the fusion of its (exact) verifiers and falsifiers. Here I would like to present a new background from which to think of the connection between truthmakers and subject matters, one with important differences from the underlying framework in which the two are usually connected. On the standard way of connecting the two notions, the subject matter of $p \vee q$, $\sigma(p \vee q)$, will be a state, namely the fusion of states verifying $p$, states verifying $q$ and of states which are themselves fusions of states falsifying $p$ and falsifying $q$ (the falsifiers for $p \vee q$). So $\sigma(p \vee q)$ contains all the states making $p \vee q$ true and false as parts. The same goes for any formula, so the subject matter of $p \wedge q$, $\sigma(p \wedge q)$, is the fusion of verifiers and falsifiers for $p \wedge q$, i.e., the fusion of states that are themselves fusions of a verifier for $p$ and a verifier for $q$ and falsifiers for $p$ and for $q$. Given that it does not matter in which order the states are fused by the associativity of fusion, then $\sigma(p \vee q) = \sigma(p \wedge q)$. In general, on this account the Boolean connectives are subject matter transparent. However, the resulting state will be impossible, as long as states verifying a given proposition $p$ and falsifying it are incompatible. And it might be thought that there is something unpalatable about saying that what we are talking about when we say "$p$" is always an impossible state that is just the fusion of all the ways that $p$ is verified and falsified. The alternative approach to truthmaker semantics I would like to provide is interesting in its own right in that while it allows to recover the same view of what subject matters are as the standard state-based view (that is, of subject matters as states that are fusions of certain other states, following the exact same structure) it also allows for an alternative account of subject matter that meets all the desiderata that the standard state-based view meets while: i) making no use of impossible states in all such cases; and ii) approximating the state-based view to classic approaches in subject matter theory where topics are identified as questions under discussion (Lewis, 1988a,b; Yablo, 2014; Yalcin, 2018; Hoek, 2022, forthcoming).

In this paper, I intend to define new notions of exact and inexact *topical* truthmaking, whose clauses, on state spaces in which different conditions are imposed, result in different notions of entailment that are both truth and aboutness preserving (as I'll argue, this means that the subject matter of the conclusion is contained in the fusion of the subject matter of the premises, plausibly meeting Francez's (2019) objections to such a project in the same fashion as Carrara, Mancini and Zhu (2022) do). This includes the Weak Kleene logic (WK), following Beall's (2016) interpretation of the undefined value as "off-topic". The motivating thought is to define (in the case of the exact semantics) topical truthmaking as a relation that holds between a state and a sentence whenever the former exactly makes true the latter *and* carries information on the whole topic of the latter[1]. The states of topical truthmaker semantics can thus be interpreted in a metaphysically innocuous way: they are simply carriers of information, which might nonetheless fail to carry positive or negative information relative to some topics.

By "carrying information on the whole of the subject matter of a sentence $S$" it is simply meant that they make true or make false every sentence that determines the topic of the sentence $S$, which if $S$ is atomic is just $S$ itself (so topical truthmaking corresponds to exact truthmaking in the atomic case) and if $S$ is not atomic, then it will be all of its components, the components of their components ... all the way down to the atoms. If a state $s$ carries information on a given topic $\mathbf{t}$[2] and on no other topic except for those that are proper parts of $\mathbf{t}$, then $s$ is an exact topical truthmaker/falsitymaker for sentences with topic

[1] Instead of just on part of it, as might be the case for instance with an exact truthmaker for $p \vee q$, which might be a state $s$ only exactly making true $p$ and neither making true nor making false $q$, therefore only carrying information concerning $\sigma(p)$ (the subject matter of $p$) and not $\sigma(q)$, even if $\sigma(p \vee q) = \sigma(p) \sqcup \sigma(q)$ (assuming the subject matter transparency of the Boolean connectives, following, among others, Fine (2017, 2020) and Berto (2022)). Carrying information on a topic $\sigma(X)$ is, thus, making true or false at least one sentence whose subject matter is part of $\sigma(X)$, carrying information on the *whole* topic is making true or false all the sentences $Y$ such that $\sigma(Y) \sqsubseteq \sigma(X)$.

[2] As is standard, I will use boldfaced notation when referring to a topic while not describing it as the topic of a sentence.

**t**, whereas if $s$ carries information on **t** but also on topics that are not proper parts of **t**, then $s$ is an *in*exact topical truthmaker or topical falsitymaker for sentences that have **t** as a topic.

As an attentive reader might have noticed[3], this intuitive picture doesn't square well with the standard understanding of subject matters in truthmaker semantics (Fine, 2020). For there subject matters are themselves *states* and then we would have to say that states of information carry information on... states of information (bloated, contradictory states of information, as a matter of fact)[4]! While it seems possible for states of information to carry information on themselves and on other states of information, this doesn't seem to be the right result: usually states of information carry information on something besides themselves. The answer I favour to this conundrum is that states carry information on how to resolve issues or questions (and therefore that we should move towards an understanding of subject matters closer to the Lewisian and inquisitive semantics tradition in this context). By making true or false sentences that share a topic with a question agents consider, states allow for agents to arrive at less and less partial (i.e. more and more informative) answers to a given question.

Having briefly characterized a new notion of *topical* truthmaking and given some philosophical motivation for it, I will now present the exact topical semantics and its corresponding notion of exact topical entailment (section 2), after that I will do the same for the inexact case (section 3), and I will show how inexact entailment in an exclusive state space yields the logic Weak Kleene, how in an exhaustive *world* space yields the Logic of Paradox and in an exclusive and exhaustive *world* space yields Classical Logic (section 4). Finally, I show how topical truthmaker semantics can naturally capture a notion of subject matter that meets plausible desiderata while not relying on identifying all subject matters with impossible states and meeting an intuitive picture of states of truthmaker semantics as states of information (section 5) before briefly summarizing and concluding (section 6).

## 2 Exact Topical Semantics

Let us start with the exact semantics. Using "$\Vdash$" for the usual notion of exact truthmaking (Fine, 2017; Fine and Jago, 2019) and "$\Vdash_t$" for the new notion of exact topical truthmaking, then we might start by giving the following clauses for atomic exact topical truthmaking and falsitymaking :

$$s \Vdash_t p \text{ iff } s \Vdash p.$$

[3]Thanks to ... for pressing me on this point and to clarify the status of the states in my semantics.

[4]NB that the subject matter of a simple atom for instance is the state that is the fusion of all the ways in which the atom is true and in which it is false. This is presumably an impossible state. Furthermore, understood as a state of information, it is not a helpful state of information to be in. I don't mean to say that this counts against the standard view, however that there is a tension here seems to me undeniable.

$$s \dashv\!\!|_t\, p \text{ iff } s \dashv\!| \, p.$$

Intuitively, by making an atom exactly true or exactly false, a state already carries information on the whole of the subject matter of that atom. So the notions of exact truthmaking and topical truthmaking coincide in the atomic case. We let as before $|p|_t+$ and $|p|_t^-$ be respectively the sets of exact topical truthmakers and falsitymakers for $p$. Given the clauses for topical truthmaking, these are respectively equal to $|p|^+$ and $|p|^-$, the sets of exact truthmakers and falsitymakers for $p$, so we can consider conditions on $|p|_t^{+/-}$ directly in terms of $|p|^{+/-}$[5]. Later we will consider spaces of states which are *exclusive*, i.e. such that any state in $|p|^+$ is incompatible with any state in $|p|^-$, that is, their fusion is an impossible state, as well on state spaces that are *exhaustive*,i.e. such that any possible state is compatible either with a member of $|p|^+$ or with a member of $|p|^-$, for any $p$.

The rest of the clauses can then be given as follows, for arbitrary sentences $A$ and $B$:

$$\begin{gathered}
s \Vdash_t \neg A \text{ iff } s \dashv\!\!|_t\, A \\
s \dashv\!\!|_t\, \neg A \text{ iff } s \Vdash_t A \\
s \Vdash_t A \wedge B \text{ iff } \exists u, v(s = u \sqcup v \wedge u \Vdash_t A \wedge v \Vdash_t B) \\
s \dashv\!\!|_t\, A \wedge B \text{ iff } \exists u, v(s = u \sqcup v \wedge u \dashv\!\!|_t\, A \wedge v \Vdash_t B) \text{ or} \\
\exists u, v(s = u \sqcup v \wedge u \Vdash_t A \wedge v \dashv\!\!|_t\, B) \text{ or } \exists u, v(s = u \sqcup v \wedge u \dashv\!\!|_t\, A \wedge v \dashv\!\!|_t\, B) \\
s \Vdash_t A \vee B \text{ iff } \exists u, v(s = u \sqcup v \wedge u \dashv\!\!|_t\, A \wedge v \Vdash_t B) \text{ or} \\
\exists u, v(s = u \sqcup v \wedge u \Vdash_t A \wedge v \dashv\!\!|_t\, B) \text{ or } \exists u, v(s = u \sqcup v \wedge u \Vdash_t A \wedge v \Vdash_t B) \\
s \dashv\!\!|_t\, A \vee B \text{ iff } \exists u, v(s = u \sqcup v \wedge u \dashv\!\!|_t\, A \wedge v \dashv\!\!|_t\, B)
\end{gathered}$$
[6]

The differences between the clauses for exact truthmaking and exact topical truthmaking are in the clauses for the falsitymakers for conjunction and the truthmakers for disjunction. While in exact truthmaker semantics it suffices, for a disjunction to be made exactly true, that a state make exactly true one of the disjuncts, in topical truthmaker semantics it must be a fusion of a state that exactly topically makes true one of the disjuncts and a state that either topically makes true or false the other disjunct (and of course, it does not matter which state does what part of the job, so there are three options). The case for the exact topical falsitymakers for the conjunction is identical.

Exact topical truthmaking comes with a notion of entailment ($\vDash_t$) distinct from exact entailment($\vDash$), where $\Gamma \vDash \varphi$ iff for any state $s$ in any model $M$, if

[5] The conditions then bubble up and apply to all formulae of the language, so if we impose a condition on $|p|^{+/-}$, it applies also for $|A|^{+/-}$ for any formula $A$ and likewise to $|A|_t^{+/-}$. The proof of this result based on the clauses presented here is left to the reader.

[6] I have been made aware that the clauses here presented for what I call topical truthmaking/falsitymaking have already been independently given by Randriamahazaka (2022). In his paper, however, different logics are focused upon, the results are generalized for infectious logics with $n$ truth-values and both strict and tolerant consequence relations. In this respect, his results are more general. On the other hand, different philosophical motivation and interpretation for the indefinite value of WK are here given. This is especially relevant as Randriamahazaka does not relate the prospects of the semantics he provides with a novel conception of subject matter in the context of a truthmaker semantics. This is, by contrast, the central focus of and main philosophical motivation for topical truthmaker semantics.

$M, s \Vdash \Gamma$ then $M, s \Vdash \varphi$, where $s \Vdash \Gamma$ iff $s \Vdash \gamma$ for all $\gamma$ such that $\gamma \in \Gamma$.[7] There might be a state exactly verifying $X$ but which does not exactly topically verify $X$. As can be surmised from the clauses for exact topical truthmaking, this corresponds in its simplest form to the case where $X = A \vee B$ for a state might exactly make $A \vee B$ true in virtue of exactly making one of the disjuncts true and not exactly making either true or false the other disjunct. Therefore the logic for exact topical entailment and exact entailment differ in an important argumentative form, valid in the latter but not in the former: Disjunction Introduction ($A \vDash A \vee B$). And this seems to be the right result given that through Disjunction Introduction, one can introduce topics that were not included in the premises (Yablo, 2014; Fine, 2020; Berto, 2022 – and one might not even possess the concepts required to grasp the new topics, that being a plausible reason why agents may fail to draw this inference, as Williamson (2000) suggests[8]).

Exact topical truthmaking is not weaker than exact entailment, however (so neither is weaker than the other). To see why, consider that $A \vee B \vDash_t ((A \wedge B) \vee (A \wedge \neg B)) \vee (\neg A \wedge B)$ (it's just a matter of unpacking the topical truthmaking clause for disjunction) but $A \vee B \nvDash ((A \wedge B) \vee (A \wedge \neg B)) \vee (\neg A \wedge B)$, since there will be a state $s$ exactly verifying $A$ that will therefore make true $A \vee B$ but that won't be the fusion of a state making true $A$ and $B$ or $A$ and $\neg B$, therefore not making true any of the disjunctions on the right of the $\vDash$[9]. This also seems to be the right result for a logic that tries to capture topic constraints. When an agent believes/knows that $p \vee q$, one way to spell out the requirement that the agent goes beyond just having an attitude towards one of the disjuncts is that they have an attitude of belief/knowledge towards the truth or falsity of the other disjunct – that is, they believe/know that the other disjunct is either true or false, and they therefore must *understand* the other disjunct. But it does not matter whether they take it to be true or take it to be false, as long as they take it to be either of them. And if it does not matter what disjunct they believe or know, then this applies for both disjuncts so that they can have the attitude towards both being true or one of them being true and the other false, as long as one of them is true. This inference licensed by exact topical entailment is the reverse side of the coin of Disjunction Introduction. The reasons for accepting one are virtually the same as the reasons for rejecting the other. It seems, then, gratuitous to reject Disjunction Introduction but not accept the inference from a disjunction to the disjunction of all three possible conjunctions of truth values in which the disjunction is true. In both cases it is simply laid down or unpacked a requirement that agents grasp the concepts involved in the disjuncts.

[7] Exact topical entailment is defined in exactly the same way, substituting $\Vdash$ for $\Vdash_t$.

[8] "Although the validity of $\vee$-introduction is closely tied to the meaning of $\vee$, a perfect logician who knows $p$ may lack the empirical concepts to grasp (understand) the other disjunct $q$. Since knowing a proposition involves grasping it, and grasping a complex proposition involves grasping its constituents, such a logician is in no position to grasp $p \vee q$, and therefore does not know $p \vee q$." in Williamson (2000, p. 283).

[9] Thanks to Greg Restall for making me notice this feature of the entailment relations for which semantics are here provided.

## 3 Inexact Topical Semantics

The logic for exact topical entailment gives us a notion of what can be derived from a state of information such that *all* the information in the state is used. Therefore from a state exactly topically making true that $A \wedge B$ it does not follow in this logic that $A$ nor that $B$, for only *part* of the information in a state making true $A \wedge B$ would be used to draw $A$ or $B$. Nonetheless, there are interesting applications for the logic of *in*exact topical truthmaking, that is the logic of what can be derived from a state of information more generally, not necessarily using all the information in it (though all the information *may* be used).

Just like in the case of the exact semantics, the inexact topical truthmaker semantics ($\rhd_t$) is an adaptation of its non-topical counterpart. In inexact truthmaker semantics ($\rhd$), states inexactly make true/false that $A$ by containing as a part a state that exactly makes true/false that $A$. The same applies for inexact topical truthmaker semantics. Starting with the atomic case, we have that:

$$s \rhd_t p \text{ iff } \exists u(u \sqsubseteq s \wedge u \Vdash_t p)$$
$$s \lhd_t p \text{ iff } \exists u(u \sqsubseteq s \wedge u \dashV_t p)$$

And for arbitrary formulae $A$ and $B$ we derive the clauses from the general requirements that $s \rhd_t A$ iff $\exists u(u \sqsubseteq s \wedge u \Vdash_t A)$ and $s \lhd_t A$ iff $\exists u(u \sqsubseteq s \wedge u \dashV_t A)$:

Atoms:

$$s \rhd_t \neg A \text{ iff } \exists u(u \sqsubseteq s \wedge s \Vdash_t \neg A) \text{ iff } \exists u(u \sqsubseteq s \wedge s \dashV_t A) \text{ iff } s \lhd_t A$$
$$s \lhd_t \neg A \text{ iff } s \rhd_t A \text{ (similar to the previous case)}$$

Conjunction:

$$s \rhd_t A \wedge B \text{ iff } \exists u(u \sqsubseteq s \wedge u \Vdash_t A \wedge B) \text{ iff } \exists v, w(u = v \sqcup w \wedge v \Vdash_t A \wedge w \Vdash_t B)$$
$$\text{iff } \exists v(v \sqsubseteq s \wedge v \Vdash_t A) \text{ and } \exists w(w \sqsubseteq s \wedge w \Vdash_t B) \text{ iff } s \rhd_t A \text{ and } s \rhd_t B$$

$$s \lhd_t A \wedge B \text{ iff } \exists u(u \sqsubseteq s \wedge u \dashV_t A \wedge B) \text{ iff } \exists u, v, w(u = v \sqcup w \wedge v \dashV_t A \wedge w \Vdash_t B)$$
$$\text{or } \exists u, v, w(u = v \sqcup w \wedge v \Vdash_t A \wedge w \dashV_t B) \text{ or}$$
$$\exists u, v, w(u = v \sqcup w \wedge v \dashV_t A \wedge w \dashV_t B)$$
$$\text{iff } s \lhd_t A \wedge s \rhd_t B \text{ or } s \lhd_t A \wedge s \lhd_t B \text{ or } s \rhd_t A \wedge s \lhd_t B$$

Disjunction:

$$s \rhd_t A \vee B \text{ iff } s \rhd_t A \wedge s \rhd_t B \text{ or } s \rhd_t A \wedge s \lhd_t B \text{ or } s \lhd_t A \wedge s \rhd_t B$$
(justification is the same to the case of inexact topical falsitymakers for the conjunction)
$$s \lhd_t A \vee B \text{ iff } s \lhd_t A \text{ and } s \lhd_t B$$
(justification is the same as for the case of the inexact topical truthmakers for conjunction)

As before, the difference between the inexact truthmaker semantics and the inexact topical truthmaker semantics comes down to the cases of the truthmakers for disjunction and the falsitymakers for conjunction. We can then define notions of inexact topical entailment ($\vDash_{\rhd_t}$)and inexact entailment ($\vDash_{\rhd}$) in the same way as their exact counterparts, where as before the former is defined on a more restricted space than the latter. As before, the simplest case in which there is an inexact truthmaker for $X$ which is not an inexact topical truthmaker for $X$ is when $X$ is of the form $A \vee B$. The logic for inexact entailment has the same validities and invalidities as FDE (Fine (2016, pp.218-219), van Fraassen (1969)). So the logic for inexact topical entailment, like the logic for Analytic Containment (AC), has the same validities as FDE minus $A \vDash_{\rhd} A \vee B$ (see Angell (1977) and Fine (2016, p. 218)). However, inexact topical entailment is not AC, as for instance $A \vee B \nvDash_{\mathsf{AC}} ((A \wedge B) \vee (A \wedge \neg B)) \vee (\neg A \wedge B)$, but $A \vee B \vDash_{\rhd_t} ((A \wedge B) \vee (A \wedge \neg B)) \vee (\neg A \wedge B)$. In fact, inexact topical entailment is just AL, developed by Oller (1999) and independently developed earlier as $S_{fde}$ by Deutsch (1977)[10]. For the same reason as before, this seems to be the right result given that we're aiming to characterize consequence relations that are both truth-preserving and aboutness preserving. This raises the question of whether AC, even though it captures a notion of content inclusion, might be too strict in one sense, as it seems that $((A \wedge B) \vee (A \wedge \neg B)) \vee (\neg A \wedge B)$ preserves the content of $A \vee B$. Besides, there are the considerations above for why it seems that if one rejects Disjunction Introduction, one should accept this inference, if one's goal is to capture constraints introduced by topics (which seems to be the case here in terms of content inclusion more generally). In our logic of inexact topical truthmaking, just like on FDE and on AC, Conjunction Elimination ($A \wedge B \vDash_{\rhd_t} A, B$) is valid.

# 4 Exclusivity and Exhaustivity

If we combine these results with the condition that the state space be *exclusive*, then we get the weak Kleene logic ($\mathsf{WK}$). If instead we impose exhaustivity and restrict our attention to *worlds* (states that are inexact topical truthmakers or falsitymakers for every sentence), we get the logic of paradox ($\mathsf{LP}$). Finally, if we impose that the state space be exclusive and exhaustive, restricting our attention to worlds again (which in this case will all be possible) then we get classical logic ($\mathsf{CL}$). Proofs for these results are given below. These last results might seem strange, for in what sense are inferences in, say, classical logic aboutness preserving? The answer is simply that they are as long as the states are required to contain information on *all* subject matters, which is the case if the world space (a state space consisting only of worlds and states included in them) is such that the information contained in them is *univocal*, that is, they each represent in regard to any given subject matter either that certain things are the case, or that they aren't the case, but not both.

[10]For discussion of this system see Ferguson (2014). For more on related systems, see Ciuni et al. (2018).

Theorem 1: *If* $|p|^+$ *and* $|p|^-$ *are exclusive, then* $\Gamma \vDash_{\rhd_t} A$ *iff* $\Gamma \vDash_{\mathsf{WK}} A$ [11].

Proof:

Left to right: Suppose $\Gamma \nvDash_{\mathsf{WK}} A$, then there is a model $\mathscr{M}$ and state $s$ such that $Vs\gamma = T$ and therefore $s \rhd_t \gamma$ for all $\gamma \in \Gamma$, but $VsA = F$ or $VsA = U$, so $s \ntriangleright_t A$ (given that the state space is exclusive in case $VsA = F$, and directly in the case of $VsA = U$). But that's just the condition for $\Gamma \nvDash_{\rhd_t} A$. Contraposing we get that if $\Gamma \vDash_{\rhd_t} A$ then $\Gamma \vDash_{\mathsf{WK}} A$.

Right to left: Suppose $\Gamma \nvDash_{\rhd_t} A$. Then there is a model $\mathscr{M}$ and a state $s$ such that $s \rhd_t \gamma$ for all $\gamma \in \Gamma$, but $s \ntriangleright_t A$. Therefore $Vs\gamma = T$ for all $\gamma \in \Gamma$ and $VsA = F$ or $VsA = U$, but only $T$ is a designated value in WK. So $\Gamma \nvDash_{\mathsf{WK}} A$. Contraposing we have that if $\Gamma \vDash_{\mathsf{WK}}$ then $\Gamma \vDash_{\rhd_t} A$. □

Theorem 2: *If* $|p|^+$ *and* $|p|^-$ *are exhaustive and* $\vDash_{\rhd_t}$ *is restricted to world states*[12] *then* $\Gamma \vDash_{\rhd_t} A$ *iff* $\Gamma \vDash_{\mathsf{LP}} A$[13].

Proof: Left to right: We prove again by contraposition. Suppose that $\Gamma \nvDash_{\mathsf{LP}} A$, then there exists a model $\mathscr{M}$ and a state $s$ such that $Vs\gamma = T$ or $Vs\gamma = B$ for all $\gamma \in \Gamma$ but $VsA = F$. Regardless of what designated value each of $\gamma$ takes, $s \rhd_t \gamma$ for all $\gamma \in \Gamma$. However, by the condition on $Vs$, we know that $s \ntriangleright_t A$. So $\Gamma \nvDash_{\rhd_t} A$. Contraposing we have that if $\Gamma \vDash_{\rhd_t} A$ then $\Gamma \vDash_{\mathsf{LP}} A$.

Right to left: Again by contraposition. Suppose that $\Gamma \nvDash_{\rhd_t} A$, then there exists a model $\mathscr{M}$ and a state $s$ such that $s \rhd_t \gamma$ for all $\gamma \in \Gamma$ but $s \ntriangleright_t A$. Automatically we have that $Vs\gamma = T$ or $Vs\gamma = B$. Since the state space is exhaustive and we are only considering world states, then $s \lhd_t A$. Given that $s \lhd_t A$ and $s \ntriangleright_t A$, then $VsA = F$. Therefore $\Gamma \nvDash_{\mathsf{LP}} A$. Contraposing we have that if $\Gamma \vDash_{\mathsf{LP}} A$ then $\Gamma \vDash_{\rhd_t} A$. □

Theorem 3: *If* $|p|^+$ *and* $|p|^-$ *are exclusive and exhaustive and* $\vDash_{\rhd_t}$ *is restricted to world states then* $\Gamma \vDash_{\rhd_t} A$ *iff* $\Gamma \vDash_{\mathsf{CL}} A$[14].

Proof:

Left to right: We prove by contraposition. Suppose that $\Gamma \nvDash_{\mathsf{CL}} A$. Then there is a model $\mathscr{M}$ and a state $s$ such that $Vs\gamma = T$ for all $\gamma \in \Gamma$ and $VsA = F$. But then $s \rhd_t \gamma$ for all $\gamma \in \Gamma$ and $s \lhd_t A$. But since the state space is exclusive, then $s \ntriangleright_t A$ and so $\Gamma \nvDash_{\rhd_t} A$. So if $\Gamma \nvDash_{\mathsf{CL}} A$ then $\Gamma \nvDash_{\rhd_t} A$. Contraposing we have that if $\Gamma \vDash_{\rhd_t} A$, then $\Gamma \vDash_{\mathsf{CL}}$.

[11] In WK there are three truth-values - $\{T, U, F\}$ -, however only $T$ is a designated value, so validity is defined as usual as truth-preservation: $\Gamma \vDash_{\mathsf{WK}} A$ if and only if in all interpretations $I$ in which all $\gamma \in \Gamma$ are such that $I(\gamma) = T$, then $I(A) = T$. For $V$ a valuation of formulae at states (in this case $s$), we let $VsA = T$ iff $s \rhd_t A$, $VsA = F$ iff $s \lhd_t A$ and $VsA = U$ iff neither $s \rhd_t A$ nor $s \lhd_t A$.

[12] That is, the states that we consider when assessing whether all the states that inexactly topically make true the premises also topically make true the conclusion must be world states.

[13] The Logic of Paradox has two designated values: $T$(rue) and $B$(oth true and false), with $F$(alse) being the only undesignated value. We let $VsA = T$ iff $s \rhd_t A$ and $s \ntriangleleft_t A$, $VsA = F$ iff $s \lhd_t A$ and $s \ntriangleright_t A$ and $VsA = B$ iff $s \rhd_t A$ and $s \lhd_t A$.

[14] Classical logic has only two truth-values which jointly apply to all formulae and never both to the same formula in the same interpretation of the same model. We let $VsA = T$ iff $s \rhd_t A$ and $VsA = F$ iff $s \lhd_t A$.

Right to left: We prove by contraposition. Suppose that $\Gamma \not\models_{\rhd_t} A$, then there is a model $\mathscr{M}$ and a state $s$ such that $s \rhd_t \gamma$ for all $\gamma \in \Gamma$ but $s \not\rhd_t A$. But since the state space is exhaustive, and we only care about what goes on in world states, then $s \lhd_t A$[15]. So $Vs\gamma = T$ for all $\gamma \in \Gamma$ and $VsA = F$, and therefore $\Gamma \not\models_{\mathsf{CL}} A$. So if $\Gamma \not\models_{\rhd_t} A$ then $\Gamma \not\models_{\mathsf{CL}} A$. Contraposing, we have that if $\Gamma \models_{\mathsf{CL}} A$ then $\Gamma \models_{\rhd_t} A$. □

# 5 Subject-Matter

Topical truthmaker semantics is more directly apt for the purposes of capturing the notion of subject matter. For starters, if one wishes, it can represent no loss (or gain) whatsoever over standard truthmaker semantics in capturing it.

We can have that the subject matter of a formula $A$ is $\sigma(A) = \bigsqcup |A|_t^+ \sqcup \bigsqcup |A|_t^-$, that is, the fusion of all its topical verifiers with all its topical falsifiers. We can then show by induction on the complexity of formulae that $\bigsqcup |A|_t^+ \sqcup \bigsqcup |A|_t^- = \bigsqcup |A|^+ \sqcup \bigsqcup |A|^-$ for any A.

Proof: The cases for atoms and $A = \neg p$ are trivial. Let $A = p \vee q$. The fusion of topical verifiers for $A$ is going to be the fusion of states that are fusions of topical verifiers for $p$ and topical verifiers for $q$, of states that are fusions of topical verifiers for $p$ and topical falsifiers for $q$ and of states that are fusions of topical falsifiers for $p$ and topical verifiers for $q$ (by the clauses for topical truthmaking for disjunction). The fusion of topical falsifiers for $A$ is the fusion of states that are themselves fusions of topical falsifiers for $p$ and for $q$. By the associativity of fusion, the fusion of topical verifiers and falsifiers for $A$ is going to be the fusion of the topical verifiers and falsifiers for $p$ with the fusion of the topical verifiers and falsifiers for $q$. But we know by two applications of the IH that the fusions of topical verifiers and falsifiers of $p$ and of $q$ are the fusions of verifiers and falsifiers of $p$ and $q$ respectively. Further, we know that the fusion of the fusion of verifiers and falsifiers of $p$ with the fusion of verifiers and falsifiers of $q$ is the fusion of verifiers and falsifiers of $p \vee q$. So we have that the fusion of topical verifiers and falsifiers of $p \vee q$ is equal to the fusion of verifiers and falsifiers of $p \vee q$. The case for conjunction is similar. □

From this result it follows that we can recover in topical truthmaker semantics exactly the same notion of subject matter as in standard truthmaker semantics. There is, however, an alternative that meets a number of plausible desiderata and makes no such heavy use of impossible states: subject matters as sets of varieties of topical truthmakers[16] and falsitymakers, where varieties

[15] Otherwise, it could be that $s$ was a state that is an inexact topical truthmaker for all the premises, but only compatible with a falsitymaker for the conclusion, without being an inexact topical falsitymaker for the conclusion itself. Adding the restriction of the consequence relation to the *worlds* in the state space ensures us that if a maximal state $s$ is compatible with $s'$, then $s' \sqsubseteq s$. And so that $s$ is also an inexact topical falsitymaker for $A$.

[16] Not to be confused with "varieties" as the term is used in category theory. Here the term is used informally to mean, roughly, the same as "kinds" or "types" of topical truthmakers, where these are also to not be read with their usual philosophically loaded meanings. A

of topical truthmakers or falsitymakers for $A$, $Vr_{t_A}$ and $Vr_{f_A}$ respectively, are sets of states topically making true $A$ and false $A$ that correspond to different terms in the clauses for topical truthmaking/falsitymaking for $A$. So where $A$ is of the form $p \vee q$, then one $Vr_{t_A}$ is going to be $|p|_t^+ \cap |q|_t^+$, another will be $|p|_t^+ \cap |q|_t^-$ and yet another will be $|p|_t^- \cap |q|_t^+$, while there is just one $Vr_{f_A}$, namely $|p|_t^- \cap |Vq^-|_t$. For the conjunction it's the reverse, there are three $Vr_{f_A}$ when $A$ is $p \wedge q$, namely $|p|_t^- \cap |q|_t^+$, $|p|_t^+ \cap |q|_t^-$ and $|p|_t^- \cap |q|_t^-$ and one $Vr_{t_A}$, namely $|p|_t^+ \cap |q|_t^+$. Finally, for the atomic case and negation there is only one variety of topical truthmaker and falsitymaker, as their clauses are not disjunctive.

This way of thinking of subject matters has benefits with respect to extant approaches in the literature as: it does not make all sentences' subject matters be impossible states (unlike the account in Fine (2020)); it is suitably more fine-grained than partition or division based accounts defined on a space of possible worlds (as in Lewis(1988a,b) and Yablo(2014)); and it is able to distinguish between contents that agents seem to have distinct propositional attitudes towards, such as *Goldbach's Conjecture* (GC) and GC $\vee\neg$GC. This is a problem for extant accounts of subject matters that employ only possible worlds (Lewis, 1988a,b; Yablo, 2014; Hawke, Özgün and Berto, 2020; Berto, 2022).

Understood this way, subject matters are sets of sets of states, just like they are sets of sets of possible worlds in classical approaches like Lewis's (1988a,b) and Yablo's (2014). Namely, for a given sentence $A$, $\sigma(A)$ will be the *partition* of the set $S$ of states that topically make true and make false $A$ into its various $Vr_{t_A}$ and $Vr_{f_A}$. Earlier we said that states carry information on topics of sentences. Now we say that topics of sentences are partitions of the set of states topically making true or false those sentences obeying certain specific requirements (*viz.* that each cell be a variety of topical truthmakers or falsitymakers for that sentence). These two claims, at first glance, don't seem to square well with one another: what does it mean for a state to carry information on a partition of a set of states of which it is a member of? Here I believe that it is helpful to think of partitions in terms of issues (Hawke, 2018) and how to resolve them. Always, in action and in conversation we can think of ourselves as responding to or addressing certain issues or questions (Hawke, 2018; Yalcin, 2018; Hoek, 2022) even if only implicitly. This makes certain distinctions between ways for things to be more salient while making others backgrounded, and we respond to the ones that are salient while ignoring the ones that are backgrounded for the moment (until a question then arises that pushes us to address that issue). States in cells of the relevant partitions will include all the information that is needed to arrive at a particular way of resolving the issue we are dealing with. This is a different way in which one can think of states carrying information on a topic $\sigma(A)$ by topically making true or false all the sentences whose topic is included in $\sigma(A)$: states allow us to resolve the issues that topics raise by

different word to get at the same thought I want to convey would be "sorts". Here I am counting on a sympathetic reader to bear with the informal usage as the precise definition comes right after. The usage of this specific term is merely suggestive.

providing all the information needed[17].

The main differences of this picture of subject matters with respect to partition-based views in standard possible worlds accounts are that now we have partitions of different sets of states for different sentences and that we are working with partial states that can themselves be in mereological relations. These two features are interrelated in how I propose is a natural way of defining subject matter inclusion in this framework: $\sigma(A) \sqsubseteq \sigma(B)$ if and only if $\forall y \forall x(x \in y \wedge y \in \sigma(A) \rightarrow \exists u \exists z(x \sqsubseteq u \wedge u \in z \wedge z \in \sigma(B)))$, that is, for all $y$, $y$ being a variety of topical verifiers or falsifiers for $A$, and therefore a member of the subject matter of $A$, and for all $x$, such that $x$ is a state topically making true or false that $A$, then if $x \in y$, $x$ is also a part of a state $u$ that is a member of $z$ for $z$ a variety of topical verifiers or falsifiers for $B$, so that $z$ is therefore a member of the subject matter of $B$.

It follows straightforwardly that this notion of parthood among subject matter obeys the minimal conditions required of a notion of parthood, namely it is a partial order (it obeys reflexivity, transitivity and anti-symmetry). Given the clauses for topical truthmaking it also follows that for any state $s$ in any variety of topical truthmakers or falsitymakers for $B$, there is a state $u$ such that $u \sqsubseteq s$ and $u$ is in a variety of topical truthmakers or falsitymakers for $A$. We have that:

Theorem 4:
$\sigma(A) \sqsubseteq \sigma(B) \rightarrow \forall x \forall y((x \in y \wedge y \in \sigma(B)) \rightarrow \exists z \exists w(z \sqsubseteq x \wedge z \in w \wedge w \in \sigma(A)))$

Proof: We start by assuming the antecedent, so that there are $A$ and $B$ such that $\sigma(A) \sqsubseteq \sigma(B)$. We then prove by induction on the complexity of $B$ that if that is the case, then every state that is a member of any member of $\sigma(B)$ has a part that is a member of a member of $\sigma(A)$. Let $B = \neg C$. By the induction hypothesis (IH), then every state in any member of $\sigma(C)$ has parts that are members of each $Vr_{t_X}$ and $Vr_{f_X}$, for $X$ any sentence such that $\sigma(X) \sqsubseteq \sigma(C)$. We know from the clauses for topical truthmaking that $\sigma(C) = \sigma(\neg C)$ (since they have the same varieties of topical truthmakers and falsitymakers no matter what $C$ is), and therefore by the transitivity of identity that $\sigma(C) = \sigma(B)$. Therefore, $\sigma(A) \sqsubseteq \sigma(C)$ and so every member of a member of $\sigma(B)$ has a part that is a member of a member of $\sigma(A)$. Let $B = C \vee D$ instead (the case for conjunction is identical and therefore omitted). By a double application of IH, we get that every state in each $Vr_{t_C}$ and each $Vr_{f_C}$ has a part that is a member of a member of $\sigma(X)$ for any $\sigma(X) \sqsubseteq \sigma(C)$ (if there aren't any, it comes out vacuously true by making the antecedent false), and that the same applies for $D$. So we again assume that $\sigma(A) \sqsubseteq \sigma(B)$. Again, if $A = B$, then the desired result follows immediately as every state is part of itself. Suppose that $\sigma(A) \sqsubseteq \sigma(B)$, $A \neq B$ and yet $\sigma(A) \not\sqsubseteq \sigma(C) \wedge \sigma(A) \not\sqsubseteq \sigma(D)$. Since $A \neq B$

[17] This happens, of course, only if a state is in the set that is partitioned, if the state is a proper part of one such state, then we're talking about a state that only carries information on a topic that is part of the initial topic or on a topic that may only overlap it (on this state). See below for topic inclusion.

then $\sigma(A) \sqsubset \sigma(B)$ (given that they will have different topical truthmakers and falsitymakers). Since $\sigma(A) \not\sqsubseteq \sigma(C)$ then there will be a state $s$ in a $Vr_{t_A}$ or $Vr_{f_A}$ such that there is no $u$ in a $Vr_{t_C}$ or $Vr_{f_C}$ such that $s \sqsubseteq u$. Similarly there won't be any state $v$ in $Vr_{t_D}$ or $Vr_{f_D}$ for which *that same state* $s$ will satisfy $s \sqsubseteq v$[18]. If $B = C \vee D$, however, then the varieties of topical truthmakers and falsitymakers of $B$ are given by those of $C$ and of $D$ (as can be surmised by the definition of subject matter and the clauses for topical truthmakikng and falsitymaking – see above for the atomic case and below for the general case) in such a way that no state is in a variety of topical truthmakers or falsitymakers for $B$ if it is not part of a state that is in a variety of topical truthmakers or falsitymakers for $C$ or for $D$. So we arrive at a contradiction. So either $\sigma(A) \sqsubseteq \sigma(C)$ or $\sigma(A) \sqsubseteq \sigma(D)$.[19] In either case, the IH applies and we know that every state that is a member of a variety of topical truthmakers or falsitymakers that is a member of, say, $\sigma(C)$ has as a part a member of a member of $\sigma(A)$. We know from the definition of subject matter and the clauses for topical truthmaking that all the states in members of $\sigma(B)$ have states in members of both $\sigma(C)$ and $\sigma(D)$ as parts. By the transitivity of the relation of parthood then every state in a member of $\sigma(B)$ has a state in a member of $\sigma(A)$ as a part whenever $\sigma(A) \sqsubseteq \sigma(B)$. $\square$

I now show that the connectives $\neg$, $\vee$ and $\wedge$ then keep being subject matter transparent. We can start by appealing to the definition of the subject matters of arbitrary sentences to get the following:

$\sigma(p) = \{|p|_t^+, |p|_t^-\}$
$\sigma(\neg A) = \{Vr^1_{t_{\neg A}}, \ldots, Vr^n_{t_{\neg A}}, Vr^1_{f_{\neg A}}, \ldots, Vr^n_{f_{\neg A}}\} = \{Vr^1_{f_A}, \ldots, Vr^n_{f_A}, Vr^1_{t_A}, \ldots, Vr^n_{t_A}\} = \sigma(A)$
$\sigma(A \wedge B) = \{Vr^1_{t_{A \wedge B}}, \ldots, Vr^n_{t_{A \wedge B}}, Vr^1_{f_{A \wedge B}}, \ldots, Vr^n_{f_{A \wedge B}}\}$[20]

We know from the clauses for topical truthmaking that the varieties of topical truthmakers/falsitymakers are at most 3, so $n \leqslant 3$. And we also know that two of the varieties of topical falsifiers for the conjunction are always varieties of topical verifiers for the corresponding disjunction (the varieties such that every state in them makes false one and only one of the conjuncts) and that the other variety of topical falsifiers of the conjunction is also a variety of topical falsifiers for the disjunction (the variety such that every state in it makes false both

[18]Note that we want it to be the case that at the *same time* $\sigma(A) \not\sqsubseteq \sigma(C)$ and $\sigma(A) \not\sqsubseteq \sigma(D)$ – there would be no contradiction in supposing just one half of the conjunction. For us to have guarantee that this is the case, there must be a state failing both conditions at the same time. It could be that $\sigma(A) \sqsubseteq \sigma(C) \vee \sigma(A) \sqsubseteq \sigma(D)$ in virtue of states being members of varieties of one when they failed to be members of varieties of the other.

[19]This effectively shows that there is nothing more to $\sigma(A \vee B)$ than $\sigma(A)$ and $\sigma(B)$, for any given $A$ and $B$, this is an important result that we will make use of later.

[20]Here's how the subject matter of the conjunction is translated to those of its conjuncts (in the most general case): $\{Vr^1_{t_A} \cap Vr^1_{t_B}, \ldots, Vr^1_{t_A} \cap Vr^n_{t_B}, \ldots, Vr^n_{t_A} \cap Vr^1_{t_B}, \ldots, Vr^n_{t_A} \cap Vr^n_{t_B}, \ldots, Vr^1_{t_A} \cap Vr^1_{f_B}, \ldots, Vr^1_{t_A} \cap Vr^n_{f_B}, \ldots, Vr^n_{t_A} \cap Vr^1_{f_B}, \ldots, Vr^n_{t_A} \cap Vr^n_{f_B}, \ldots, Vr^1_{f_A} \cap Vr^1_{t_B}, \ldots, Vr^1_{f_A} \cap Vr^n_{t_B}, \ldots, Vr^n_{f_A} \cap Vr^1_{t_B}, \ldots, Vr^n_{f_A} \cap Vr^n_{t_B}, \ldots, Vr^1_{f_A} \cap Vr^1_{f_B}, \ldots, Vr^1_{f_A} \cap Vr^n_{f_B}, \ldots, Vr^n_{f_A} \cap Vr^1_{f_B}, \ldots, Vr^n_{f_A} \cap Vr^n_{f_B}\}$.

conjuncts). Finally, we know that the variety of topical truthmakers for the conjunction is a variety of topical truthmakers for the disjunction as well. So every variety of topical truthmakers and topical falsitymakers for a conjunction finds a match in the corresponding disjunction. We have then that:

$\sigma(A \wedge B) = \sigma(A \vee B)$

To prove transparency, we will need to assume exhaustivity. But as it was seen, this assumption only amounts to accepting that one can form world states (that can be both possible or impossible in the usual sense of representing a way things can be or a way things cannot be, if one does not add exclusivity), that is, states that for each sentence either topically make it true or topically make it false (or both). In fact, this has by itself no consequence on the logic for which a semantics is provided, but only when we restrict our attention to what goes on in the world states instead of on every state (as can be seen earlier from the equivalence between inexact topical entailment on certain restricted state spaces in the case of $\mathsf{LP}$ and $\mathsf{CL}$). In fact, by running the exact same proof, we can show that in an exclusive and exhaustive space, inexact topical entailment still corresponds to $\mathsf{WK}$[21]. So even though there is an extra commitment of defining subject matters in this way, I take it that it is a very minimal assumption that is independently plausible. At the intuitive level, and thinking again of states as carriers of information on given topics, then this amounts to the assumption that a state carrying positive or negative information about a topic can be joined with a state carrying positive or negative information about any other topics. Even though we might be interested, for certain uses, in capturing a notion of information in which information on topics cannot be conjoined at will (say, for instance, that they could only be joined together if there existed a salient or relevant topic of which the topics were both part, as when questions are grouped under disciplines), I don't think that in general there should be a ban on the possibility of joining these pieces of information together once they exist. In a sense, they come for free once the basic pieces of information are already given.

We move on, then, to prove that the topic of a conjunction or disjunction is just the fusion of the topics of their constituents. If we define fusion, $\sqcup$, of the subject matters of sentences as the least upper bound of those subject matters in terms of the relation of parthood $\sqsubseteq$ as defined above, then:

Theorem 5: $\sigma(A \vee B) = \sigma(A \wedge B) = \sigma(A) \sqcup \sigma(B)$.

Proof: We have that $\sigma(A \vee B) = \sigma(A \wedge B) = \{Vr^1_{t_{A\wedge B}}, \ldots, Vr^n_{t_{A\wedge B}}, Vr^1_{f_{A\wedge B}}, \ldots, Vr^n_{f_{A\wedge B}}\}$. We show that i) $\sigma(A) \sqsubseteq \sigma(A \vee B) = \sigma(A \wedge B)$ and $\sigma(B) \sqsubseteq \sigma(A \vee B) = \sigma(A \wedge B)$ and ii) for all $\sigma(X)$ such that $\sigma(A) \sqsubseteq \sigma(X)$ and $\sigma(B) \sqsubseteq \sigma(X)$ then $\sigma(A \vee B) \sqsubseteq \sigma(X)$. We start by proving i). We do this for $A$ as the case for $B$ is identical. We do it by induction on $A$. Let $A$ be an atom. Then $\sigma(A) = \{|A|^+_t, |A|^-_t\}$. It follows for arbitrary $A$ by exhaustivity that any

[21] Even though he thinks of exhaustivity in different terms, Randriamahazaka's(2022) semantics for weak Kleene are in terms of an exhaustive state space. Regardless, the present paper highlights how an exhaustivity requirement is neither here nor there w.r.t. to providing a semantics for weak Kleene as long as exclusivity is enforced.

state $s$ that topically makes $A$ true or topically makes $A$ false is compatible either with a topical truthmaker or a topical falsitymaker for $B$. We prove that $s$ is part of a state in $Vr_{t_{A\vee B}}$ or $Vr_{f_{A\vee B}}$. Let's suppose that $s$ topically makes $A$ true. Then it will be part of a state topically making true $A$ and either topically making true or false $B$, that is part of a state in $|A|_t^+ \cap |B|_t^+$ or $|A|_t^+ \cap |B|_t^-$[22], so $s$ will be part of a state in a variety of topical truthmakers for $A \vee B$ and a variety of topical truthmakers for $A \wedge B$ or a variety of topical falsitymakers for $A \wedge B$. Suppose instead that $s$ makes $A$ false, then by exhaustivity $s$ is compatible either with a state in $|B|_t^+$(a $Vr_{t_B}$) or a state in $|B|_t^-$ (a $Vr_{f_B}$), and therefore will be part of a state in either $|A|_t^- \cap |B|_t^+$ (a $|A|_t^- \cap Vr_{t_B}$) or $|A|_t^- \cap |B|_t^-$ (a $|A|_t^- \cap Vr_{f_B}$). So $s$ will be part of a state that is either in a variety of topical truthmakers for $A \vee B$ or a variety of topical falsitymakers for $A \vee B$ and always in varieties of topical falsitymakers for $A \wedge B$. So regardless $s$ is part of a state that is in a variety of topical truthmakers or falsitymakers for $A \vee B$ and since $s$ was an arbitrary state in a variety of topical truthmakers or falsitymakers for $A$, then we can conclude that all states in all varieties of topical truthmakers or falsitymakers for $A$ are part of a state in a variety of topical truthmakers or falsitymakers for $A \vee B$, i.e. $\sigma(A) \sqsubseteq \sigma(A \vee B)$. The case for $A = \neg C$ is straightforward. We apply the IH and get that for all varieties of topical truthmakers and falsitymakers for $C$, there is a state in each of them that is part of a state in a variety of topical truthmakers or falsitymakers for $C \vee B$ and $C \wedge B$. But we know that $\sigma(C) = \sigma(\neg C)$, so the varieties of topical falsitymakers of one are the varieties of topical truthmakers of the other, and vice-versa. So automatically we get the result we wanted that $\sigma(A) \sqsubseteq \sigma(A \vee B)$. The last case is for when $A = C \vee D$. By a double application of the IH, we have that $\sigma(C) \sqsubseteq \sigma(C \vee D)$ and $\sigma(D) \sqsubseteq \sigma(C \vee D)$ (and similarly for the subject matter of the disjunctions $C \vee B$ and $D \vee B$). So we know that all states in varieties of topical truthmakers or falsitymakers for $C$ are parts of states that are in varieties of topical truthmakers or falsitymakers for $C \vee D$ (and the same for $D$ instead of $C$). Further, as proven just above, every state in a variety of topical truthmakers or falsitymakers for $C \vee D$ has a state in a variety of topical truthmakers or topical falsitymakers for $C$ as a part (and similarly for $D$). By exhaustivity, each of those states topically making true or false that $C \vee D$ is compatible either with a state topically making true that $B$ or topically making false that $B$, which is the same as saying that their fusion exists. But their fusion, by the topical truthmaking and falsitymaking clauses for disjunction is always going to be a state either topically making true that $B$ or topically making false that $B$. So for any state $s$ in each $Vr_{t_A}$ and in each $Vr_{f_A}$, there is a state $u$ in a $Vr_{t_{A\vee B}}$ or in a $Vr_{f_{A\vee B}}$ such that $s \sqsubseteq u$. So again $\sigma(A) \sqsubseteq \sigma(A \vee B)$. So regardless of what form $A$ takes, $\sigma(A) \sqsubseteq \sigma(A \vee B)$.

We now prove ii). We know that every state in $A$ and in $B$ is part of a state in varieties of topical truthmakers and falsitymakers of $X$ (by the definition of

[22]In fact, this only happens if we assume that $B$ is itself atomic. The more general case is instead: supposing $s \in |A|_t^+$, then there exists a state $s'$ such that $s \sqsubseteq s'$ and $s' = s \sqcup u$ for a state $u \in Vr_{t_B}$ or $u \in Vr_{f_B}$. Nothing would change if instead we had presented the general case.

parthood among subject matters and the assumption that $\sigma(A) \sqsubseteq \sigma(X)$). We also know from i) that $\sigma(A) \sqsubseteq \sigma(A \vee B)$ (and same for $B$), so every state $s$ that is in a variety of topical truthmakers or falsitymakers for $A$ is part of a state that is in a variety of topical truthmakers or falsitymakers for $A \vee B$ (the same for $B$). We know, however, that any state $s$ that is in a variety of topical truthmakers or falsitymakers for $A$ is part of a state in a variety of topical truthmakers or falsitymakers for $X$, and so is a state $u$ such that $u$ is in a variety of topical truthmakers or falsitymakers for $B$. But then their fusion, $s \sqcup u$, is also part of a state that is in a variety of topical truthmakers or falsitymakers for $X$. But all states that are in varieties of topical truthmakers and falsitymakers for $A \vee B$ are fusions of the form $s \sqcup u$, for every combination of $s$ and $u$, as given by the topical truthmaking and falsitymaking clauses for disjunction. So all states that are in varieties of topical truthmakers and falsitymakers for $A \vee B$ are part of a state in a variety of topical truthmakers or falsitymakers for $X$.[23]

So $\sigma(A \vee B) = \sigma(A \wedge B) = \sigma(A) \sqcup \sigma(B)$. □

Finally, there is a natural way of adapting this view of subject matters so that they correspond to a notion of subject matters as questions under discussion. The varieties of topical truthmakers and falsitymakers are just sets of states, which are themselves often partial, i.e. they fail to topically make true or false various sentences. So these sets are not yet propositions as they are not upwards-closed with respect to the relation of parthood among states. However we can easily recover propositions from them. A proposition would be upwards closed with respect to the relation of parthood among states, and would be given by *truthmaking* conditions, namely, it would be the set of all states having an exact topical truthmaker as a part. So, say that $\varphi$ is the proposition corresponding to the sentence $p$, then:

$$\varphi = \{s : s \rhd_t p\}$$

We can then identify subject matters instead with sets of propositions (so understood), where each proposition corresponds to a direct answer to a question, replacing in the definition of subject matters the varieties of topical truthmakers and falsitymakers for the respective propositions[24]. Consider the simple case of the subject matters of two atomic propositions (where, in effect, we ignore the complications introduced by varieties), $\varphi$ and $\psi$ and their conjunction and disjunction:

[23] That is, the result that $\sigma(A \vee B)$ is the least upper bound of the relation of parthood among subject matters comes directly from how the states in varieties of topical truthmakers or falsitymakers for $A \vee B$ are themselves fusions of states in members of $\sigma(A)$ and in $\sigma(B)$, and for this notion we just assume that fusion of states *is* the least upper bound of the relation of parthood among states.

[24] Given a sentence $A$ and a variety $Vr^i_{t_A}$ of topical truthmaker for it, there will be a corresponding proposition, defined as follows: $\varphi[Vr^i_{t_A}] = \{s : \exists u(u \sqsubseteq s \wedge u \in Vr^i_{t_A})\}$. It is with propositions like this that one can then in general define subject matters in terms of propositions instead of varieties themselves.

$$\sigma(\varphi) = \{\varphi, \neg\varphi\}$$
$$\sigma(\psi) = \{\psi, \neg\psi\}$$
$$\sigma(\varphi \vee \psi) = \sigma(\varphi \wedge \psi) = \{\varphi \cap \psi, \varphi \cap \neg\psi, \neg\varphi \cap \psi, \neg\varphi \cap \neg\psi\}$$ [25]

# 6 Conclusion

Truthmaker semantics has been gaining traction in recent years. And its application to the notion of topic has been one of its most significant developments. Even though the standard truthmaker semantics inspired state-based view of subject matters is perfectly serviceable as it is, there are minor details of it that seem unpalatable to some.

Here I have presented an alternative semantics that is both sound and complete in its inexact version to $AL/S_{fde}$, in an exclusive state space with respect to Weak Kleene, in an exhaustive world space to the Logic of Paradox and in an exhaustive and exclusive world space to Classical Logic. It allows us as well to capture the notion of subject matter in a way that makes connections with classic ways of understanding the notion, such as that of Lewis (1988a,b) and Yablo (2014). Finally, we saw how in this view of subject matters, impossible states are not necessarily part of the picture.

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[25] Note that given the informal description above, we have that propositions are always formed from the varieties of topical *truthmakers* for a sentence. This aligns with the usual understanding of propositions as truth-conditions. So in all three of these equations we're appealing to the fact that the set of all inexact truthmakers for $\neg A$ is the same as the set of all inexact falsitymakers for $A$, for any $A$.

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