Sets as fusions of materially equivalent rigid embodiments

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Large parts of the mathematical domain can be reduced to talk of sets and set-theoretic relations. This reduction has represented a gain in terms of providing large portions of mathematics with an epistemically more secure footing, as they were now founded in set theory.

But we should also want more: mathematics should be founded in a theory that is both epistemically and *metaphysically* on more secure footing. Developments in set theory, such as the axiomatic tradition () surely go a long way with regard to the former end: the definitions and axioms of set theory are rigorously presented and its theorems carefully derived. However, one might have some misgivings about the ontological presuppositions of set theory, and with regard to this the mathematical developments of set theory (as opposed to its philosophical developments) do not seem to help us much.

A notable instance of the, on the face of it unseemly, ontological commitments of set theory is the relation between singletons and their single members. On the one hand it seems that they should be distinct entities if we accept the principle of identity of indiscernibles, for they have different properties: for instance the singleton $\{a\}$ has a as a member, whereas (by the axiom of foundation) even if a is a set, it does not have itself as a member. On the other hand, the informal characterizations of sets as "many taken as one" do not seem to shed light on what makes up the distinction between the member and its singleton. A given single entity taken as one seems to just be that same entity[1]. Another source of worry is the nature of the empty set. After all, what is it to collect no entities into one?

Given worries such as these, one might wish to take the reductionist project further and to reduce sets to other entities. Lewis (1991) attempts a reduction of set theory to classical extensional mereology (CEM) with plural quantification. However, he (begrudgingly) accepts a primitive "singleton function", that is, a function that attributes to any urelement or set its singleton, which he takes to be a wholly disjoint entity from its single member[2] and to be an atomic

[1]See Lewis(1991) for other worries about taking the notion of singleton to be well-understood, as well as for criticisms of some ways out proposed for how to account for the nature of singletons.

[2]That is, an entity having no part in common with its single member.

class (i.e. something that has members). The rest of set theory is, for Lewis, simply CEM as all classes will just be fusions of singletons, where the subset relation is therefore the restriction to classes of the usual relation of parthood. If we follow Lewis, however, we still have to deal with the mystery of how it is that singletons and their sole members relate and how they can be completely disjoint entities. Given that all singletons, for Lewis, would be mereologically atomic classes, then it's not at least obvious why an object (urelement or set) a is related to one such atom $\{a\}$ instead of any other such that $\{a\}$ is a's singleton.

Stressing the second kind of worry, Lewis (1991) is forced to take the empty set to not be itself a class, but rather an individual. This is so as the empty set is not itself a fusion of singletons. Lewis's notion of a set has, therefore, to be disjunctive: a given S is a set if and only if it is a class that has a singleton (as opposed to proper classes) *or* it is the empty set (Lewis, 1991, 18).

What individual is the empty set, for Lewis? He takes it to be arbitrary, insofar as we have a way of always selecting an individual whenever there exist any individuals. For this reason, Lewis makes the arbitrary choice of taking the empty set to be the fusion of all individuals.

This conception of the empty set has three immediate and related problems: (i) it makes the empty set (and therefore all pure sets) have contingent properties; (ii) it disagrees with certain seemingly innocuous mathematical claims; and (iii) it makes the definition of set disjunctive in an arbitrary way.

Let's start with the first worry. Call the fusion of all individuals the *universe*. Both Lewis (1986) and anyone who accepts the metaphysical underpinnings of quantified modal logic with variable domains, will then have to say that in each world there are distinct individuals which are, in that world, the universe. If we accept a strong recombination principle (as Lewis does), then we will even have the consequence that any individual (or a counterpart thereof) is the empty set in some world. But different objects will have different properties, by a simple contraposition of the principle of the identity of indiscernibles. It follows, then, that the empty set has a number of contingent properties in each possible world. But this seems to fly in the face of how we usually talk, and of how we do mathematics. Why should the identity of the object we introduce into our theory (of which we usually say nothing except that it is a memberless set) care about what contingent individuals there are in the world? If the empty set is indeed different objects in worlds with different domains, why does it happen that we only study and care about the properties that all such individuals share, and, besides, only those that can be expressed in terms of the usual set-theoretical predicates? For instance, if only Timmy existed in the world, and Timmy is thin, would thereby the empty set be thin? To my mind, this is a category mistake.

This very quickly relates to the second worry. It seems to be an innocuous set theoretical statement that no pure set is a set of urelements, and vice-versa. But if Lewis's account is correct, then this would be (necessarily) false for all such sets save for the empty set. The empty set is always going to be identical to an individual, and so necessarily the pure sets (save for the empty set) will all be sets of urelements (though different urelements at different worlds). Effectively, for any individual o, we seem to be able to make a distinction between $\{o\}$ and $\{\emptyset\}$, but for one such individual, we would be incorrect in making that distinction. Further, what individual that is will vary from world to world. We would then have it that even though a set is a set of urelements in this world, it is a pure set in another world (and vice-versa for worlds whose domain is a proper superset of ours). This raises the worry that in a world where o is the empty set, $\{\emptyset\}$ and $\{o\}$ seem to be intensionally distinct but extensionally equivalent sets. This seems wrong, and the reason is that the empty set should not be o.

In his reductionist project, Lewis effectively seems to discard what we usually call the pure sets, in how they are usually understood. We should, on the other hand, take their queer nature at face value. There is something unique about the empty set in that it seems to be a set obtained from nothing. This pervades the whole pure hierarchy of sets insofar as they are all constructed from the empty set. The theory to be defended will keep the possibility of making sets "out of thin air" in this way, and I won't argue *for* such a possibility. Rather, if there is anything objectionable to this process, then I believe we should be suspicious of set theory's presuppositions to begin with.

Moving on, then to the third and last worry. Given that it was arbitrary which individual one would take to be the empty set, then it seems equally arbitrary to say that *this* individual is a set instead of *that* one. So our arbitrariness in picking out the empty set infects the definition of set itself. So besides being a disjunctive definition, the second part of the definition is ad-hoc, for which individual we take to be the empty set is entirely trivial. For instance, if o is the empty set in the actual world, then it will be false that, say, $\{o\} \in o$. But for another individual similar to o, say o', it seemingly won't simply be false that $\{o\} \in o'$ but rather this won't even be truth-evaluable as there will be a category mistake: o' is not a set, so it doesn't make sense to say that things are or are not members of *it*[3]. But how could the distinction between a merely false claim and a category mistake depend simply on our arbitrary decision to have one be the empty set and not the other? To press this worry, how could a sentence be a category mistake if there is a possible world (given a strong recombination principle) in which it is true? It seems implausible that we need such a cumbersome and arbitrary definition of set, even if we care about reducing set theory. Relevantly, it seems we ought to keep the feeling that the empty set is a set in the same way as all other sets – it just so happens that it is *the* set that has no members and of which no set is a proper subset.

One of the primary motivations to attempt a reduction of set theory was precisely to account for the mysterious notion of a singleton, so it seems we haven't completely achieved our goals if we accept a primitive singleton function.

[3]The reader might protest saying that individuals are related to sets by the membership relation, and therefore it would be the right result that such a sentence is simply false. But to say this would be to say that just because, say, the predicate "feels" relates certain animals to emotions, as in "Mabel feels happy", then it isn't a category mistake to say "Happiness feels Mabel", as Mabel and happiness keep being the relata.

Furthermore, such a reduction will not be complete if we treat the pure sets as "second-class citizens" by taking them to really be sets of urelements. Staying within the remits of mereology, and given that one would wish to clarify the connection between singletons and their sole members then one natural thought, with some intuitive pull to it (see (Fine, 1999, 2010)), is to take the members of sets to be parts of sets[4].

Informed by Fine's (1999) theory of rigid embodiments, Caplan, Tillman and Reeder (2010) attempt a reduction of set theory to a non-classical mereology in which sets are identified as rigid embodiments (objects that exist insofar as given object(s) (the rigid embodiment's material part(s)) instantiate a given property or relation (the formal part of the rigid embodiment)). Caplan *et al.*'s (2010) view (which I will refer to as the "CTR view/account" in later discussion) is successful in its attempt to reduce set theory to mereology while at the same time dispelling the mystery of singletons and the empty set – if, that is, one is (as I am) happy to endorse Fine's theory of rigid embodiments. Still, I believe one can do better.

Fine's theory of rigid embodiments, in its standard form, deviates from classical extensional mereology in not validating even weak supplementation. This principle is, however, taken by some to be constitutive of the very notion of parthood (Simons, 1986; Varzi, 2009). Given that sets are themselves extensional, one might think that the fragment of the underlying mereology that accounts for sets should itself be extensional. Below I show how the CTR view also violates weak supplementation for sets (that is, if we ignore other wholes and restrict our quantifiers to range only over rigid embodiments that are sets), and elaborate on why we should want our mereology when applied to sets to itself be extensional.

Further, in the framework of Fine's mereology there are three fundamental forms of composition (Fine, 2021): the usual notion of fusion, plausibly taken to be the least upper bound of the relation of parthood; the principle of rigid embodiment; and the principle of variable embodiment. It is then an open question, not yet developed in detail as far as I am aware, how these three modes of composition work together[5]. Given that fusion restricted to objects that are not rigid or variable embodiments is itself a plausibly extensional mode of composition, a natural thought is to consider sets to be fusions of certain rigid embodiments[6]. Very roughly put, the principle of rigid embodiment would account for why members would be parts of sets, whereas the usual relation of fusion would allow us to regain extensionality. It is this thought that I here pursue, attempting a different reduction of set theory to mereology: one that

[4]This does not immediately help with the problem of the empty set. As will become clearer later, accepting the empty set requires accepting a seemingly magical process of building many entities out of nothing.

[5]Fine (1999) as well as Jacinto and Cotnoir (2019) tell us how timeless and temporary parts of wholes interact, but not about fusion. As we will see, combining fusion with these forms of composition does not lead to many surprises, so the omission is no fault on their part.

[6]In what follows I will only consider rigid as opposed to variable embodiments as a set's members don't change overtime, whereas variable embodiments' parts may change over time.

deviates as little as possible from CEM when it comes to sets (i.e., if we restrict our quantifiers to sets and their parts), while also dispelling the mysteries of the connection between singletons and their sole members and of the empty set.

In section 1, I motivate the general thought that members of sets are parts of sets, and how in that way, as opposed to extant attempts in the literature, the mystery of the connection between singletons and their sole members gets dispelled. In section 2, I both present the CTR view and elaborate on how one might improve on it, arguing for why the mereology of sets should be extensional. In section 3, I informally present my own reductive account of set theory, where sets are identified with fusions of materially equivalent rigid embodiments (i.e., rigid embodiments that have the same material part(s)). In section 4, I then provide a formal characterization of the view and further derive the axioms of Z set theory, following the presentation in Boolos (1971) and Caplan *et al.* (2010). Finally, in section 5 I wrap up and conclude.

1 Taking members of sets to be parts of sets

It is now standard to think that the mereological analogue in set theory of the relation of parthood is the relation of subsethood (Lewis, 1991), instead of the relation of membership. For one, it agrees nicely with a well-studied and (at least) mathematically natural mereological structure: classical extensional mereology, which is isomorphic to a Boolean algebra with the bottom element removed (see Cotnoir and Varzi (2021)).

But I think that there is also a very natural, perhaps pre-theoretic, inclination to consider the members of sets as parts of sets (see (Fine (1999, 2010)) for an instance of a mereological system in which the members of sets are *the* parts of sets). In what follows I would like to consider the prospects of taking this thought seriously and I attempt a reduction of set theory to a non-classical mereology in which the members of sets are parts of sets (albeit not the *only* parts of sets).

This clashes with Lewis's perspective and what seems to be the consensus since the publication of *Parts of Classes* on the mereology of sets, namely that only the subsets of a set are the parts of a set. My opposition to this trend does not stem merely from a primitive appeal to the intuition that members should be parts of sets. Rather, taking the hypothesis that the members of sets are parts of sets seriously is, I suggest, the only viable way of addressing the mysterious connection between singletons and their sole members. To see why, I'll consider several attempts to characterize the connection between singletons and their sole members and how metaphysically (un)satisfactory they turn out to be, showing further how their failures directly relate to the generalized version of the objection raised by Lewis (1986) to Magical Ersatzism[7].

[7] Magical Ersatzism is the (pejorative) term employed by Lewis to refer to theories on the metaphysics of possible worlds that take possible worlds to be primitive abstract entities, which he attributes to philosophers like van Inwagen (1986) and Stalnaker (1976).

As I aim to show, if the singletons are themselves not atomic (i.e. are not entities not having any proper parts) but display mereological structure of the appropriate kind, then the puzzlements evaporate. While this is so, immediate puzzles make themselves apparent when the hypothesis is considered of taking the singletons to be complex. Properties and relations can play an important role in solving these, namely in showing what the parts of singletons other than the sole individuals they each contain might be. It is for this reason that a framework like Fine's, where properties and relations play precisely such a mereological role, will be taken to be particularly helpful. This is the strategy. Let's move on then to the first step in the argument: the evaluation of theories taking singletons to be atomic.

One first way in which atomic singletons have proven to be unsatisfactory is that they presuppose a primitive connection between the singletons and their sole members. Lewis (1991, p. 29 and following) presents a strong case against precisely this primitive notion of a singleton function (i.e., a function that maps each individual to its singleton), which he begrudgingly accepts. Such a primitive connection between abstract simples (the singletons) and concrete individuals seems to be in no better standing, metaphysically speaking, than what Lewis called the "selection relation" between the magical ersatzists' elements (which play the role of worlds in their theory) and the happenings in our (Lewisian-style) universe. About such a relation, Lewis asks us: why is it that the happenings in our universe pick out this "element" (i.e. this atomic possible world) and not some other? Similarly, we can ask, why does the singleton function map each entity to the singleton that it does and not to any other entity? Any other atom would do, so long as we ensure that the element would then be a member of the singleton.

Instead of accepting such a primitive notion, one could then give up on trying to give a characterization of the singleton function altogether and instead, like Lewis (1991, pp. 45-54) ends up suggesting as a way forward, adopt a structuralist approach. On this way out, all the sentences of our theory of sets where the word "singleton" appears would be ramseyfied out, so that no interpretation would be given to the term "singleton". Rather, it would only be claimed that reality is structured in such a way that there is at least one function obeying the structural features necessary to make all such sentences (i.e., those where "singleton" occurs) true together.

Sadly for the proponent of a view according to which singletons are mereologically atomic, here too there is a parallel with the criticism that Lewis (1986) directs at magical ersatzists, and that indeed he seems to some extent to still agree with in the case of sets. The criticism goes as follows. While when doing semantics or mathematics, the structuralist approach may work perfectly well, it does not seem satisfactory when one is aiming to provide a metaphysical story of the matters at issue. Here, remember, we are precisely trying to ascertain whether we can give mathematics a more secure metaphysical footing, by providing an account of what sets are that dispels some misgivings about singletons and the empty set.

The reason why the structuralist approach is not satisfactory in this context is that it was precisely the question of what the singletons *are* that we wanted to get an answer to. To be told that they are whatever fits certain structural constraints will not be satisfactory, for we already are in possession of the functional concept of a singleton, what we want is to be given entities that can fulfil it (Van Inwagen, 1986), if any can. The answer, therefore, presupposes that some entity can, but that's exactly what's at issue. Of course, the structuralist will claim that there is no fact of the matter beyond these structural constraints as to what singletons (and other mathematical entities in general) are, as we learn from Benacerraf's identification problem (Benacerraf, 1965). Applied to sets instead of natural numbers, we would have that sets can be reduced in a number of different ways, so we face a problem of saying which reduction should be preferred. But an opponent won't have to claim uniqueness, it might very well be that multiple reductions are in good standing. What the structuralist's opponent would claim is simply that for any such reduction to be satisfactory, it must be the case that there is a connection between the entities that are the members and the entities that are the singletons besides simply the latter being the singletons of the former, for otherwise we still have no independent explanation for why some entities are the singletons of other entities. Further, the opponent might claim, the functional concept of a singleton seems to impose a great demand on the size and structure of reality, so that the demands that set theory would impose are far from trivial. Namely, it seems that given the order of infinity of the class of sets that it is possible to construct, and how at each stage it is possible to construct the singletons of the sets created at the previous stage, it would seem that there would be more singletons than concrete objects[8]. So it would be impossible to combine this version of structuralism with a nominalist stance (as (Horsten, 2022) suggests one should go about, if a structuralist is to be able to successfully avoid Benacerraf-Field's epistemic problem itself (Benacerraf, 1973; Field, 1989)). Just like a structuralist defence of magical ersatzism fails to tell us what entities play the role of the ways the world could have been, a structuralist defence of atomic singletons fails to tell us what entities the singletons are, and therefore what entities play the role of being such that only one thing is a member of them and (if Lewis is right) of being a part of every class that has them as a subclass.

Aside from saying that the singletons are atomic and that there is a primitive unexplained connection between singletons and their sole members; or going structuralist and claim that there isn't a specifiable favoured function from individuals to singletons, one could also, as some have done, follow Quine and identify some of the singletons with the individuals themselves (what have been called the "Quine-atoms"). This move would eliminate the mysterious connection at its root: the singletons would just be the individuals[9]. While there is

[8]Lewis's (1991) example is to consider that any configuration of matter across spacetime points is a different individual. Even so, we get at most two to the continuum-many concrete individuals, whereas the order of infinity of the class of sets is much greater than this.

[9]Technically on this account the singletons need not be atomic, for they will only be atomic if the individuals are atomic. However I'm still considering this view as according to it the singletons would only have as parts its members and its parts, and the parts of the member

some merit to this strategy, as there does not seem to be much intuitive difference between one and the other, there are obvious difficulties lurking in the surroundings to this hypothesis if it applies to non-individuals [10]. Suppose that $\emptyset = \{\emptyset\}$, then $\emptyset \in \emptyset$, but the empty set has no members, so we have a contradiction. Given this result, you might then worry that the resulting account of sets would also be unnecessarily disjunctive. The singletons of individuals would be identical to the individuals, but what about all other singletons? Does the singleton function behave in a different way in the other cases? Or does it work in the same way, and simply yields this result for individuals? Regardless of what one might say in such cases, I believe one shouldn't even begin to identify sets with their singletons, for analogous reasons to why I object to Lewis's notion of the empty set. To say that "Compincha is a member of Compincha" seems meaningless, Compincha (a cat) isn't a class or set, so it doesn't make sense to say that anything (including Compincha) is a member of himself. Since Compincha is a cat, is therefore {Compincha} a cat? An affirmative answer to this question seems to me to be clearly misguided, and yet it would be correct if we identified the two. Set theorists don't study cats or the like, even when they consider sets of urelements. Rather, they may study sets whose members are cats.

We have seen then that taking singletons to be atomic, whether distinguishable from their sole members or not, led to mystery in all three proposals. I propose, then, to distinguish the singletons from their members, and sets overall from their members, rejecting Quine's move. I aim to do so, of course, in a way that does not resort to a primitive unexplained connection between entities or to a structuralist escape route. Rather, just like Lewis famously explains the relation between Lewisian possible worlds (maximally concrete and disconnected wholes) and what they represent via mereological relations, here I'll aim to explain the relation between singletons and their sole members and more generally between sets and their members via the mereological structure of sets.

The argument for why members should be proper parts of sets makes itself: singletons can't be atomic for otherwise we can't explain the connection between member and singleton, just like we can't explain the relation between atomic abstract possible worlds and our universe (i.e. Lewisian-style possible world). Likewise, singletons can't simply be identical to their members. So we conclude that members should be proper parts of sets, and more generally proper parts of sets.

There are further reasons to think that no other such relation between member and singleton could do the job. I present them again resorting to a move inspired by Lewis's (1986) arguments against Magical Ersatzism, namely by van Inwagen's (1986) application of it to sets, also mentioned in Lewis (1991). Suppose that we explain why S is the singleton of a by appealing to a relation R,

that are not identical to the member don't play a role, so we can pretend all individuals are atomic.

[10] Lewis (1991, p. 42) gives a summary of the difficulties, here I mention one. In fact, Quine () defines individuals by means of them being identical to their singleton, so this theory was never meant to apply more generally.

and suppose that a is not part of S. Since clearly the singleton of a is distinct from the singleton of a' for a' a perfect duplicate of a (but, read, such that $a \neq a'$), then the relation R cannot hold between a and S solely based on a's intrinsic properties or S's intrinsic properties (i.e. properties that they have and would continue to have were each of them to be the only objects in the world). It cannot be therefore an internal relation. So suppose instead that R is an external relation, and therefore that it depends on the properties of a and S taken together. Still, there is a necessary connection: presumably S is necessarily a's singleton. But since a and S are disjoint entities, if we accept (for the sake of argument) a strong recombination principle, then S could exist without a, so how can there exist such a necessary connection? This, it seems to me, is enough to throw suspicion, on Humean grounds, on the connection between member and singleton *if* the former is not part of the latter. But if we take a member to be a proper part of its corresponding singleton, then there is no such worry, the relation of parthood guarantees a necessary connection between S and a: whenever the singleton exists, there exists the member[11].

A further, tentative positive reason to consider that the members of sets are parts of sets comes from how the subset relation is naturally interpreted as a parthood relation on sets (Lewis, 1991; Cotnoir and Varzi, 2021) and from the axiom of set extensionality. The axiom of set extensionality is usually written as $\forall S \forall T(\forall z(z \in S \iff z \in T) \iff S = T)$, with the first two quantifiers restricted to sets. But it can also be written in terms of the subset relation $\forall S \forall T((S \subseteq T \wedge T \subseteq S) \iff S = T)$, case in which the axiom of extensionality can be understood as an instance of mutual parthood between S and T (again, supposing that $\subseteq$ corresponds to $\sqsubseteq$). But if this is so, then just as naturally one can think of the first formulation of the axiom as a formulation of proper parthood extensionality $\forall S \forall T(\forall z(z \sqsubset S \iff z \sqsubset T) \iff S = T)$, again with the first two quantifiers restricted to sets.

Going for one or the other view has a big impact on what claims of parthood we make in set theory. For instance, if we claim that only the members of sets and their proper parts are proper parts of sets, we will have to say that any set is only a proper part of a set that contains it as a member. So for instance the set $\{\{a\}\}$ will contain $\{a\}$ as a proper part, but not $\emptyset$ as a proper part, even though $\emptyset \subseteq \{\{a\}\}$. But obviously $\emptyset \neq \{\{a\}\}$, and it seems equally natural, as it has seemed to many, to take $\subseteq$ to be a relation of parthood. So we would get that $\emptyset \sqsubseteq \{\{a\}\}$ and that $\emptyset \neq \{\{a\}\}$, but at the same time $\emptyset \not\sqsubset \{\{a\}\}$. So something would seem to be amiss.

I take it, instead, that we should accept both of the natural claims and accept that *both* members and proper subsets of sets (and perhaps other entities) are proper parts of sets. It is notable, in this respect, that the extensionality axiom can also be rewritten in this form appealing to a notion of proper subsethood:

[11] An attentive reader will notice that simply saying this will not dispel the worry about there being worlds where a exists but not $\{a\}$. In what follows I try to assuage this worry by arguing that: (i) necessarily every object has properties which it necessarily instantiates; (ii) for each instantiation of a property by a plurality of objects, there is the rigid embodiment of those objects in the plurality *qua* having the property.

$\forall S \forall T \forall U((U \subset S \iff U \subset T) \iff S = T)$. Here I don't claim that this is a strong positive reason. Rather, this is a defeater for there being anything special about subsets. If one's reason to take subsets to be parts of sets is the close analogy between set extensionality and the parthood extensionality, then clearly there is also such an analogy with extensionality expressed in terms of membership.

1.1 A Lasso Theory?

When considering various alternative views on the mereology of sets, Lewis considers what he calls a lasso hypothesis, that is, a view according to which sets have as parts their members plus an extra part, a binding element – a lasso. He then goes on to criticize lasso theory on the grounds that demonstrably (Lewis, 1991, p. 44) if the sets are fusions of their members and a further part, then the lasso would have to be different for every set. The reason for this goes roughly as follows. We suppose CEM and therefore extensionality. Suppose we have two individuals a and b and suppose that $\{a\}$ and $\{b\}$ are such that $\{a\} = a + l$[12] and $\{b\} = b + l$. We will then also have the set $\{a, b\}$, suppose this is $a + b + m$. Then $\{\{a, b\}, a\}$ will be $a + b + l + m$ and similarly $\{\{a, b\}, b\}$ will be $a + b + l + m$. But these are the same whole, whereas we have the same set, which is a contradiction.

Lewis claims, then, that just as initially there was a mystery as to what the singletons were and what connections there were between them and their sole members, this mystery would just be replaced by a new mystery – that of what the lassos are, which would be these new mysterious entities in one-one correspondence with the sets.

The lasso, as envisioned by Lewis, seems to be a further individual that is added to the members to form the set. But this need not be so. For instance on Caplan et al.'s (2010) view, sets are rigid embodiments, and we could say that the lasso is the property of "having some attribute or other". In this case, the connection between the lasso and the members is supposed to not be mysterious – it's just instantiation of a plural distributive property, that is, a property of some things sucb that each of them individually has it.

So if instead of an individual, we state that the further parts of sets besides their members are properties plurally instantiated by the objects that are members of the set, then the objections just raised by Lewis cease to hold, as there won't be an individual added by fusion in the case of all sets to their members to form the respective sets, which is especially significant since the relation between properties and the objects that instantiate them is not one of fusion. The required way in which the "lasso" — that is, the further parts of the set besides the members — are added to the sets ought not to be by the way of fusion[13]. This serves as a further motivation for why we turn now to a mereological theory in which properties figure as parts of objects and in a way

[12] For + a pairwise relation of fusion.
[13] Lewis (1991), of course, rejects such "unmereological" forms of composition.

that respects the unique way in which properties and the individuals that instantiate them are combined, namely to the Finean (1999) hylomorphic theory of rigid embodiments.

2 Rigid Embodiments and Global Properties

Following a venerable Aristotelian tradition, Fine (1999) distinguishes between various material objects' form and matter, or formal and material parts, the former corresponding to their properties or relations to other objects and the latter to the object itself, abstracted away from these properties and relations.

According to Fine, most objects, or at least most concrete objects, have both formal and material parts. As an example, Fine mentions a ham sandwich. In order for a particular ham sandwich to start existing, it does not suffice that the slices of bread and ham it is made of start existing. Rather, it seems that it is only once they are arranged in a certain way, with the slice of ham inserted between the slices of bread, that the ham sandwich starts existing, and only insofar as they keep being arranged in that way the ham sandwich keeps existing. One speaks properly, in this sense, when one says that one is *making* a sandwich, for before one's act there is still no sandwich in existence.

On Fine's theory, the ham sandwich would correspond to an object made up of its constituents — the ham and bread slices -– in the relation of the ham being between the two bread slices. Where / represents a primitive principle of rigid embodiment, b_1, b_2 and h_1 stand respectively for the bread slices and for the ham slice, and B for the relation of "betweenness", we would have that the ham sandwich is the rigid embodiment $b_1, h_1, b_2/B$. That is, the ham sandwich is the slices of bread and ham *qua* the latter being between the former.

I take it to be very intuitive that various objects only exist insofar as given objects instantiate certain properties or certain other objects are related to them in given ways, like the ham sandwich. Therefore, this conception will be accepted in what follows as the best explanation for such cases.

Caplan *et al.* (2010) have provided a theory of sets, and therefore of singletons, based on Fine's (1999) theory of rigid embodiments, as just briefly presented, and on a theory of group properties defined using tools from plural logic.

Just like ordinarily in first order logic we are able to quantify over individuals, Boolos (1984) has maintained that in natural languages we also quantify over pluralities of individuals, including in cases where paraphrases in terms of the corresponding singular quantifications are unnatural or leave out part of the content. To take an example, consider the sentence "the stones of Stonehenge form a round-ish shape", and consider how none of the rocks by itself forms the given shape. Further, Boolos has influentially argued that this feature of how one quantifies in several natural languages can be given a perspicuous account in terms of a plural logic, which just like regular first order logic contains constants, individual variables and quantifiers, as well as both logical and non-logical predicates, but which adds both plural variables and quantifiers, as well as the logical predicate "is one of", so that "y is one of xx" is a formula of the language of first order quantified plural logic where y and xx are respectively free individual and plural variables. For readability purposes, instead of "xx" we might use the locution "x's" in for instance "y is one of the x's" in what follows.

In order to avoid arity issues that would immediately arise, one can define with resources from plural logic and predicate abstraction plural properties, that is, properties that are shared by a plurality of objects. For this, we need to introduce further predicates that apply not to individuals or tuples of individuals, but rather to pluralities directly. Common examples are non-distributive properties like *form a circle* (let us refer to this property as F), so that from $F(xx)$ we can't derive that there is a y that is one of the x's such that $F(y)$, that is, such that y forms a circle (intuitively, it is only the multiple objects that form a circle, not any of them taken individually). The property of forming a circle would be a plural property of the x's. However, a property need not be non-distributive to be a plural property, it need only apply to the plurality, whether or not it then distributes to its constituents. For instance *Self-identical (mb)*, (where mb is the plurality such that Mabel is one of those things in it and Kami is one of those things in it as well, and nothing else is) would be the plural identity property applied to the plurality formed by Mabel and Kami, even if both Mabel and Kami also possess the property of being self-identical on their own *qua* individuals. Plural properties would be obtained in this way from formulae containing plural variables (either bounded or free) via predicate abstraction, so that whenever such predicates are satisfied, there is a corresponding plural property instantiated by the respective pluralities. In what follows all the rigid embodiments that will be considered will have plural properties as their formal parts, not relations.

The reason for this is that typically relations introduce considerations having to do with the order by which the terms of the relation are related. In Fine's (1999) original theory, rigid embodiments can be aptly characterized as tuples of individuals and relations (as Jacinto and Cotnoir (2019) do). But since the purpose here is to adapt Fine's theory to account for sets, we won't be taking into account rigid embodiments where the order of the material parts matters. It is for this reason that we restrict our attention to rigid embodiments of the form xx/P where xx is a plurality and P is a plural property.

It is a theory of plural properties sketched out in this way that will be accepted in what follows, and it is a theory in the lines of which Caplan *et al.* (2010) appeal to when developing their reductive view of the nature of sets.

2.1 The CTR view

Based on both frameworks just presented, as well as the Armstrongian (1991) intuition that for given objects to form a set they must in some sense "be a one", Caplan, Tillman and Reeder (2010) go on to claim that sets just correspond to the rigid embodiments where the members of the corresponding sets are the material parts and where the formal part just is the plural property of *having* *some attribute or other*, which, since all objects have this property, will apply to all material parts of the rigid embodiment. We will henceforth call this the CTR view. Caplan *et al.* then move on to identify the empty set with the limit case of a rigid embodiment with the relevant formal part but no material parts, its singleton as the set having the empty set as its only material part and so on for all the other pure sets.

Despite of inheriting some advantages from Fine's theory of rigid embodiments, and therefore having a property as a part of sets and having the members of sets as parts of sets, Caplan *et al.*'s view faces a major obstacle: it violates, like Fine's mereological framework more generally, weak supplementation.

To see how their view leads to a failure of weak supplementation, consider again the case of the empty set and its singleton. While the former only has a formal part whereas the latter has both a formal and material part, they are all identical to one and the same entity, *viz.* the property *having some attribute or other*, as the empty set will be /*having some attribute or other* and its singleton will be (/*having some attribute or other*)/*having some attribute or other*[14]. The singleton of the empty set will have the empty set as its proper part but no proper part disjoint from the empty set, for in the case of the singleton the property *having some attribute or other* applies to a rigid embodiment, but the rigid embodiment itself is just composed of the property *having some attribute or other*, nothing else. Here note that it is because they accept the same formal part for every set that we can derive the failure of weak supplementation.

In what follows I aim to proceed from where Caplan *et al.* left off and, with suitable changes, put forward a view that: i) recovers the same axioms of Z set theory; ii) is compatible with the principle of weak supplementation and proper parthood extensionality for sets; and iii) according to which the members of sets are parts of sets.

Here the reader might wonder: why should our underlying mereology be itself extensional? If we have doubts about set theory's metaphysical underpinnings and want to reduce sets to other entities, then we should pay utmost attention that we do recover the usual set theoretical principles. But why go beyond that and require that our underlying entities themselves be extensional or obey some of the demands imposed by the nature of sets?

My reply starts from the fact that any differences between sets that do not amount to their membership seem to be superfluous given set-extensionality: why would two sets be the same if and only if they have the same members if the entities that the sets are can differ while having the same parts that are their respective members? Plausibly, if that was the case then we could have two different sets with the same members but merely changing in whatever

[14]In some passages, Caplan *et al.* (2010) refer to the empty set as just being the property *having some attribute or other* itself. However if that were so, then how could one form the empty set's singleton? Rigid embodiments are formed by objects instantiating certain properties or relations, but then the empty set would not be an object, but rather a property, so how could one form the rigid embodiment $\emptyset$/*having some attribute or other*? For this reason, I prefer to interpret their view as stating that the empty set is a rigid embodiment (therefore an object) with no material part.

else makes up their parts. Further, since to alleviate the worry that there is a mysterious connection between singletons and their sole members, we agreed with Fine (1999, 2010) and Caplan et al. (2010) that the members of sets are parts of sets, then any distinction between sets should come down to a difference between the members of sets, and therefore the parts of sets that are its members. Therefore all distinctions between sets will be distinctions between parts of sets.

Is this enough motivation for weak supplementation? It seems to be only enough for parthood and proper parthood extensionality.

In my view sets always have proper parts (their members). Given that we only care about distinctions between sets that amount to distinctions between the parts of sets that are their members, why suppose that there are more parts of sets than their members? And if there are no more parts of sets, then weak supplementation fails: $\{a\}$ will have no proper parts that are not parts of a. But this seems to be exactly what led us to the problem of singletons: there seems to be nothing "making up the difference" between the singleton and its sole member. If we had no problem saying that a and $\{a\}$ are distinct, that the former is a proper part of the latter but that there is nothing else making up the difference between the two, then there would be no mystery of what the connection is between member and singleton. But there *does* seem to be something mysterious going on here, after all, that's why some (Simons (1986), Varzi (2009)) even take weak supplementation to be constitutive of the meaning of "proper part".

As it will be seen, following Jacinto and Cotnoir (2019), I follow a presentation of Fine's (1999) theory of rigid embodiments where weak supplementation holds in general. What I believe *might* not hold in general is proper parthood extensionality. However, that comes from cases where the same components arranged in different ways might give rise to distinct wholes. But, risking repeating myself, sets are extensional and there should be no differences between sets merely based on differences between the arrangements of their parts. On the other hand, if relations are themselves best understood in an orderless, "neutral" way (Fine, 2000), then perhaps even in this case we could uphold extensionality.

3 Fusions of Rigid Embodiments and Cumulative Hierarchy

The guiding intuition is then that Caplan *et al.* go wrong in having selected a particular "candidate" to be a distinguished formal part of all sets. Rather, I hold that different sets ought to have different formal parts, and that, starting from given individuals (or no individuals, in the case of the pure sets), sets can be constructed hierarchically. The view to be defended is that sets are fusions of materially-equivalent rigid embodiments. For that, we will need to first have in place some crucial notions, namely those of: material-equivalence among rigid embodiments; and of fusion of rigid embodiments.

First, we can say that given rigid embodiments are materially equivalent whenever they have the same material parts, so that if xx/P and yy/Q are rigid embodiments, they are materially equivalent if and only if each of the x's is one of the y's and each of the y's is one of the x's (that is, if $\forall u(u \prec xx \iff u \prec yy)$). The rigid embodiment of the slices of bread and ham *qua* being concrete is materially equivalent to the rigid embodiment of the slices of bread and ham *qua* being self-identical.

Having a notion of material equivalence between rigid embodiments in place, we can group rigid embodiments in terms of what material parts they contain that mirrors what members given sets have. Caplan *et al.*'s view can then be thought as selecting for each such group the rigid embodiment having as the formal part *having some attribute or other* to serve as the set having as members the material parts of the rigid embodiments that are materially equivalent. Instead, I do not want to select any of the materially equivalent rigid embodiments to play any such special role. For this reason, I take sets to be the fusion of all rigid embodiments sharing a material part and move on to define the operation of fusion on rigid embodiments.

Before I introduce the notion of fusion, it will be important to introduce the notion of stages. Following Jacinto and Cotnoir (2019), which themselves are inspired by Fine's (2005) conception of the hierarchy of formation of sets and of abstractions, we can think of rigid embodiments as being formed in stages. So we have a stage 0 where we have no rigid embodiments but only individuals formed by the horizontal process of composition, we have a stage 1 where we have all the objects of stage 0, plus all the rigid embodiments formed of pluralities of objects given at stage 0, plus all possible fusions of objects given up to stage 1. And so on for stages 2, 3, etc. We then have an ordinal stage k such that every other stage is below k but there is no stage i immediately below k (that is, a stage k such that there is no stage i that is below k and for which there is no stage j such that $i < j < k$).

In what follows I assume stage-relative unrestricted fusion. So I say that for any x's given at a stage i, which I express by saying $xx@i$(sometimes I say level instead of stage), y is the fusion of the x's, as follows $F_{xx@i}y$ (borrowing notation from Cotnoir and Varzi (2021)). y is itself given at i, i.e. $y@i$.

Like Caplan et al. (2010), I restrict myself to rigid embodiments having a *distributive* plural property as a formal part, so that if $(xx/P)@i$ then $P(xx)$ and if x is one of the xx, then $P(x)$ and $x@i$.

Finally, I will be appealing to an overarching notion of parthood $\sqsubseteq$, as well as two other cognate notions of parthood: $\lll$ between material and formal parts of rigid embodiments and the rigid embodiments of which they are part; and $\trianglelefteq$ for parts of wholes resulting from the typical (horizontal) process of fusion. It will be important in what follows to evaluate how these relate. In trying to compatibilize a hylomorphic mereology with the axioms of CEM restricted for sets, I intend this to be read as $\sqsubseteq$ behaving like parthood in CEM when restricted to sets. A central feature of the present account is that material parts of rigid embodiments that are themselves parts of fusions of rigid embodiments will be parts of those fusions of rigid embodiments, by transitivity of $\sqsubseteq$, which is proven below. To prove this result I will need to appeal to the fact that if the xx are part ($\lll$) of a rigid embodiment, then so is each x that is one of the x's. The thought is that to say that the x's are the material parts of a given rigid embodiment is *just to say* that each of the x's is a material part of the rigid embodiment, and this is so even if none of them could have been a material part of such a rigid embodiment in isolation (as it seems to happen in cases where the formal part is non-distributive and/or relational).

With these definitions in place, as well as an appeal to Leibniz's Law and a principle stating that all individuals have properties, one can derive the axioms of $\mathscr{Z}$ set theory. This is proven in the formal appendix below.